New Polynomials and Numbers Associated with Fractional Poisson Probability Distribution

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Abstract

Generalizations of Bell polynomials, Bell numbers, and Stirling numbers of the second kind have been introduced and their generating functions were evaluated.

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1 Fractional Poisson Probability Distribution

Bell polynomials, Bell numbers [1] and Stirling numbers of the second kind [2]-[4] are related to the well known Poisson probability distribution. Recently, the fractional Poisson probability distribution has been developed [5], and some of its applications have been implemented in [6]. The fractional Poisson probability distribution is a natural model which captures long memory impact on the counting process.

Here we present and explore generating functions for new polynomials and numbers related to the fractional Poisson probability distribution. Those new polynomials and numbers are: fractional Bell polynomials, fractional Bell numbers and fractional Stirling numbers of the second kind (see, [6]).

The fractional Poisson probability distribution $P_\mu(n, t)$ of arriving n items ($n = 0, 1, 2, ...$) by time t is given by [5]

$$P_\mu(n, t) = \frac{(\nu t^\mu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\nu t^\mu)^k}{\Gamma(\mu(k+n)+1))}, \qquad 0 < \mu \le 1, \qquad (1)$$

where the parameter ν has physical dimension $[\nu] = \sec^{-\mu}$ and the gamma function $\Gamma(\mu)$ has the familiar representation $\Gamma(\mu) = \int_0^\infty dt e^{-t} t^{\mu-1}$, $\text{Re}\mu > 0$. At $\mu = 1$ Eq.(1) is transformed into the well known equation for the standard

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Poisson probability distribution with substitution $\nu \to \overline{\nu}$, where $\overline{\nu}$ is the rate of arrivals of the standard Poisson process with physical dimension $\overline{\nu} = \sec^{-1}$.

2 Fractional Bell polynomials and Bell numbers

Based on the fractional Poisson probability distribution Eq.(1), we introduce a new generalization of Bell polynomials

$$B_\mu(x,m) = \sum_{n=0}^{\infty} n^m \frac{x^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-x)^k}{\Gamma(\mu(k+n)+1))}, \qquad B_\mu(x,0) = 1, \quad (2)$$

where the parameter μ is $0 < \mu \leq 1$. We will call $B_\mu(x,m)$ as fractional Bell polynomials. Polynomials $B_\mu(x,m)$ are related to the well-known Bell polynomials [1] $B(x,m)$ by $B_\mu(x,m)|_{\mu=1} = B(x,m) = e^{-x} \sum_{n=0}^{\infty} n^m \frac{x^n}{n!}$. From Eq.(2) we come to the formula for numbers $B_\mu(m)$, which we call the fractional Bell numbers

$$B_\mu(m) = B_\mu(x,m)|_{x=1} = \sum_{n=0}^{\infty} \frac{n^m}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-1)^k}{\Gamma(\mu(k+n)+1))}. \qquad (3)$$

Now we focus on the general definitions given by Eqs.(2) and (3) to find the generating functions of polynomials $B_\mu(x,m)$ and numbers $B_\mu(m)$. Let us introduce the generating function $F_\mu(s,x)$ of polynomials $B_\mu(x,m)$ as

$$F_\mu(s,x) = \sum_{m=0}^{\infty} \frac{s^m}{m!} B_\mu(x,m). \qquad (4)$$

To find an explicit equation for $F_\mu(s,x)$, we substitute Eq.(2) into Eq.(4) and evaluate the sum over m. As a result we have

$$F_\mu(s,x) = E_\mu(x(e^s - 1)), \qquad (5)$$

where $E_\mu(z)$ is the Mittag-Leffler function given by its power series [7]

$$E_\mu(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\mu m + 1)}. \qquad (6)$$

If we put $x = 1$ in Eq.(5), then we immediately come to the generating function $\mathcal{B}_\mu(s)$ of the fractional Bell numbers $B_\mu(m)$

$$\mathcal{B}_\mu(s) = \sum_{m=0}^{\infty} \frac{s^m}{m!} B_\mu(m) = E_\mu(e^s - 1). \qquad (7)$$

In the case of $\mu = 1$, Eq.(5) turns into the equation for the generating function of the Bell polynomials $F_1(s,x) = \exp\{x(e^s - 1)\}$, while Eq.(7) reads $\mathcal{B}_1(s) = \sum_{m=0}^{\infty} \frac{s^m}{m!} B_1(m) = \exp(e^s - 1)$, and we come to the equation for the Bell numbers generating function.

3 Fractional Stirling numbers of the second kind

Now we introduce the fractional generalization of the Stirling numbers of the second kind $S_\mu(m,l)$ by means of equation

$$B_\mu(x,m) = \sum_{l=0}^{m} S_\mu(m,l) x^l, \tag{8}$$

where $B_\mu(x,m)$ is a fractional generalization of Bell polynomials given by Eq.(2) and the parameter μ is $0 < \mu \leq 1$. At $\mu = 1$, Eq.(8) defines integers $S(m,l) = S_\mu(m,l)|_{\mu=1}$, which are called Stirling numbers of the second kind. At $x = 1$, when fractional Bell polynomials $B_\mu(x,m)$ become fractional Bell numbers, $B_\mu(m) = B_\mu(x,m)|_{x=1}$, Eq.(8) gives us the equation to express fractional Bell numbers in terms of fractional Stirling numbers of the second kind $B_\mu(m) = \sum_{l=0}^{m} S_\mu(m,l)$.

To find $S_\mu(m,l)$ we transform the right-hand side of Eq.(2) as follows

$$B_\mu(x,m) = \sum_{l=0}^{\infty} \frac{x^l}{\Gamma(\mu l + 1)} \sum_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m, \tag{9}$$

where the notation $\binom{l}{n} = \frac{l!}{n!(l-n)!}$ has been introduced. By comparing Eq.(8) and Eq.(9) we conclude that the fractional Stirling numbers $S_\mu(m,l)$ are given by

$$S_\mu(m,l) = \frac{1}{\Gamma(\mu l + 1)} \sum_{n=0}^{l} (-1)^{l-n} \binom{l}{n} n^m, \tag{10}$$

$$S_\mu(m,0) = \delta_{m,0}, \qquad S_\mu(m,l) = 0, \quad l = m+1, \quad m+2, ...$$

As an example, Table 1 presents a few fractional Stirling numbers of the second kind.

m\l	1	2	3	4	5	6
1	$\frac{1}{\Gamma(\mu+1)}$					
2	$\frac{1}{\Gamma(\mu+1)}$	$\frac{2}{\Gamma(2\mu+1)}$				
3	$\frac{1}{\Gamma(\mu+1)}$	$\frac{6}{\Gamma(2\mu+1)}$	$\frac{6}{\Gamma(3\mu+1)}$			
4	$\frac{1}{\Gamma(\mu+1)}$	$\frac{14}{\Gamma(2\mu+1)}$	$\frac{36}{\Gamma(3\mu+1)}$	$\frac{24}{\Gamma(4\mu+1)}$		
5	$\frac{1}{\Gamma(\mu+1)}$	$\frac{30}{\Gamma(2\mu+1)}$	$\frac{150}{\Gamma(3\mu+1)}$	$\frac{240}{\Gamma(4\mu+1)}$	$\frac{120}{\Gamma(5\mu+1)}$	
6	$\frac{1}{\Gamma(\mu+1)}$	$\frac{62}{\Gamma(2\mu+1)}$	$\frac{540}{\Gamma(3\mu+1)}$	$\frac{1560}{\Gamma(4\mu+1)}$	$\frac{1800}{\Gamma(5\mu+1)}$	$\frac{720}{\Gamma(6\mu+1)}$

Table 1. *Fractional Stirling numbers of the second kind $S_\mu(m,l)$ $(0 < \mu \leq 1)$*

To find a generating function of the fractional Stirling numbers $S_\mu(m, l)$, let's substitute $B_\mu(x, m)$ from Eq.(8) into the definition given by Eq.(4). Hence, we have

$$F_\mu(s, x) = \sum_{m=0}^{\infty} \frac{s^m}{m!} \left(\sum_{l=0}^{m} S_\mu(m, l) x^l \right) = \sum_{l=0}^{\infty} \left(\sum_{m=l}^{\infty} S_\mu(m, l) \frac{s^m}{m!} \right) x^l. \qquad (11)$$

On the other hand, from Eq.(5), we have $F_\mu(s, x) = \sum_{l=0}^{\infty} \frac{(e^s - 1)^l}{\Gamma(\mu l + 1)} x^l$. Upon comparing this equation and Eq.(11) we introduce the generating function $\mathcal{G}_\mu(s, l)$

$$\mathcal{G}_\mu(s, l) = \sum_{m=l}^{\infty} S_\mu(m, l) \frac{s^m}{m!} = \frac{(e^s - 1)^l}{\Gamma(\mu l + 1)}, \qquad l = 0, 1, 2, ... \qquad (12)$$

In addition, we introduce the generating function $\mathcal{F}_\mu(s, t)$ defined by

$$\mathcal{F}_\mu(s, t) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} S_\mu(m, l) \frac{s^m t^l}{m!} = \sum_{l=0}^{\infty} \frac{t^l (e^s - 1)^l}{\Gamma(\mu l + 1)} = E_\mu(t(e^s - 1)). \qquad (13)$$

As a special case $\mu = 1$, equations (12) and (13) include the well-know generating function equations for the standard Stirling numbers of the second kind $S(m, l)$ (for example, see Eqs.(2.17) and (2.18) in Ref.[4]).

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